LETTER TO THE EDITOR

Purely gravito-magnetic vacuum space-times

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Abstract. It is shown that there are no vacuum space-times (with or without cosmological constant) for which the Weyl-tensor is purely gravito-magnetic with respect to a normal and timelike congruence of observers.

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1. Introduction

Non-conformally flat space-times for which the metric is an exact solution of the Einstein field equations

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab} \tag{1}$$

and in which there exists a family of observers with 4-velocity u^a $(u_a u^a = -1)$ such that the gravito-electric part of the Weyl-tensor vanishes,

$$E_{ab} \equiv C_{abcd}u^b u^d = 0, \tag{2}$$

are called purely gravito-magnetic space-times. The remaining gravito-magnetic part of the Weyl-tensor,

$$H_{ab} \equiv C^*_{abcd}u^b u^d, \tag{3}$$

has no Newtonian analogue and its role in the dynamics of the gravitational field is not very clear, apart from the fact that it is associated with gravitational radiation [1, 2, 3]. Although purely gravito-magnetic space-times are the subject of some intensive current research [4, 5, 6, 7, 8, 9, 10], only a handfull of models with a reasonable matter source is known, a situation which is in stark contrast to the purely gravito-electric space-times, for which wide and physically important classes of examples exist. This is particularly true for the vacuum solutions $T_{ab} = 0$, where for example all the static vacua are purely gravito-electric, while no purely gravito-magnetic solutions are known at all! This has lead some researchers to conjecture that purely gravito-magnetic vacua do not exist [6, 8], but this so far has only been proved in the special cases where the Petrov type is D [6] or where the timelike congruence u^a is normal and shear-free [4]. In the present letter it is shown that the conjecture is true provided that the congruence is normal only .

2. Dynamical equations

For a vacuum purely gravito-magnetic space-time in which the timelike congruence u^a is normal, one of the constraint equations (the "divergence of $\mathbf{E}$" equation, see for example [8]) guarantees that the shear tensor σ_{ab} commutes with H_{ab} (note that it makes sense to talk about *the* kinematical quantities, as these space-times are necessarily of Petrov-type I [6] and, as can be easily seen, the timelike congruence is then uniquely defined). As both tensors are orthogonal to the timelike congruence it follows that an orthonormal tetrad (with $\mathbf{u} = \mathbf{e}^0$) exists in which σ_{ab} and H_{ab} are diagonal. From now on I will follow the notations and conventions of the orthonormal tetrad formalism [11], with the exception of the coefficients $n_{\alpha\alpha}$ being redefined as follows:

$$n_{11} = (n_2 + n_3)/2,\ n_{22} = (n_3 + n_1)/2,\ n_{33} = (n_1 + n_2)/2 \tag{4}$$

As the system of equations is SO(3)-invariant, each triplet of equations will be represented by a single equation (the others being obtained by cyclic permutation of the indices). The vanishing of the gravito-electric part of the Weyl-tensor can then be expressed by the 9 equations

$$E_{11} \equiv -\partial_0\theta_1 - \theta_1^2 + \partial_1\dot{u}_1 + \dot{u}_1^2 - \dot{u}_2(a_2 - n_{13}) - \dot{u}_3(a_3 + n_{12}) + \frac{1}{3}\Lambda = 0 \tag{5}$$

$$E_{12} \equiv \partial_2\dot{u}_1 + \dot{u}_2(\dot{u}_1 + n_{23} + a_1) + \frac{1}{2}\dot{u}_3 n_2 + \Omega_3(\theta_2 - \theta_1) = 0 \tag{6}$$

$$E_{21} \equiv \partial_1\dot{u}_2 + \dot{u}_1(\dot{u}_2 - n_{13} + a_2) - \frac{1}{2}\dot{u}_3 n_1 + \Omega_3(\theta_2 - \theta_1) = 0 \tag{7}$$

The vanishing of the off-diagonal components of H_{ab} on the other hand leads to

$$\begin{aligned} H_{12} \equiv & -\partial_0(n_{12} + a_3) - \partial_1\Omega_2 - \theta_1(n_{12} + a_3 + \dot{u}_3) \\ & +\Omega_1(n_{13} - a_2) - \Omega_2(n_{23} - a_1 + \dot{u}_1) + \frac{1}{2}\Omega_3(n_1 - n_2) = 0 \end{aligned} \tag{8}$$

$$\begin{aligned} H_{21} \equiv & -\partial_0(n_{12} - a_3) - \partial_2\Omega_1 - \theta_2(n_{12} - a_3 - \dot{u}_3) \\ & +\Omega_1(n_{13} + a_2 - \dot{u}_2) - \Omega_2(n_{23} + a_1) + \frac{1}{2}\Omega_3(n_1 - n_2) = 0 \end{aligned} \tag{9}$$

It might strike as odd that the off-diagonal components of E_{ab} and H_{ab} are listed as independent equations, as both are symmetric tensors. This is the price one has to pay (or perhaps the benefit which is obtained?) by steering away from the covariant approach: the symmetry of E_{ab} and H_{ab} is now guaranteed by the Jacobi-identities. This becomes clear when we use the previous expressions to obtain (a) evolution equations for θ_α, a_α, $n_{\alpha\beta}$ and (b) expressions for the spatial gradients $\partial_\alpha\dot{u}_\beta$ ($\alpha \neq \beta$) of the acceleration. Of the 16 Jacobi-identities and the 6 (0α) and ($\alpha\beta$) ($\alpha \neq \beta$) components of the field equations only 15 algebraically independent equations remain. These equations can be written as evolution equations for the coefficients n_α,

$$\partial_0 n_1 = 2\partial_1\Omega_1 + 2\dot{u}_1\Omega_1 + 4(\Omega_2 n_{13} - \Omega_3 n_{12}) - n_1\theta_1 - n_2(\theta_1 - \theta_3) - n_3(\theta_1 - \theta_2) \tag{10}$$

and as expressions for the spatial gradients of θ_α and n_α:

$$\partial_1\theta_2 = (\theta_2 - \theta_1)(n_{23} + a_1) \tag{11}$$

$$\partial_2\theta_1 = (\theta_2 - \theta_1)(n_{13} - a_2) \tag{12}$$

$$\partial_1 n_2 = -2\partial_2(n_{12} + a_3) - 4n_{12}(n_{13} - a_2) - 2n_{23}n_1 + 2a_1 n_2 \tag{13}$$

$$\partial_2 n_1 = -2\partial_1(n_{12} - a_3) + 4n_{12}(n_{23} + a_1) + 2n_{13}n_2 + 2a_2 n_1 \tag{14}$$

Finally, the relation between the diagonal components of H_{ab} and the rotation coefficients is given by

$$2H_{11} = n_2\theta_3 + n_3\theta_2 - (n_2 + n_3)\theta_1 \tag{15}$$

3. Propagating the Einstein equations

The equations of the previous section allow us now to propagate the diagonal components of the Einstein equations along $\mathbf{u}$. While the (00) component is an identity, the $(\alpha\alpha)$ components yield

$$\begin{aligned} 2\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3 + \partial_3 n_{12} - \partial_2 n_{13} \\ = 2(n_{23}^2 + a_1^2 + a_2^2 + a_3^2) - 2(a_2 n_{13} + a_3 n_{12}) - \frac{1}{2}n_2 n_3 \\ -\theta_1(\theta_2 + \theta_3) + \frac{2}{3}\Lambda \end{aligned} \tag{16}$$

The trace of these equations is given by

$$\begin{aligned} 4(\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3) = 2(n_{12}^2 + n_{23}^2 + n_{13}^2) + 6(a_1^2 + a_2^2 + a_3^2) \\ -\frac{1}{2}(n_1 n_2 + n_2 n_3 + n_3 n_1) - 2(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1) \end{aligned} \tag{17}$$

Acting now with the ∂_0 operator on (16) one can eliminate the second order derivatives of the coefficients a_α by using the evolution equations of the previous section together with the commutator relations $[\partial_0,\ \partial_\alpha]a_\alpha$. What complicates matters at this stage is the introduction of various second order *spatial* derivatives of the Ω_α (which describes the rotation of the spatial triad with respect to a Fermi-propagated frame). A small miracle however ensures (a) that these derivatives always appear under the form of $[\partial_\alpha,\ \partial_\beta]\Omega_\gamma$ commutators and hence can be eliminated and (b) that the remaining first order derivatives of Ω_α all cancel out. Furthermore the resulting equation contains the a_α derivatives only in the "divergence of $\mathbf{a}$" form, which is known from (17). The final result is a remarkably simple triplet of equations:

$$(n_2 - n_3)n_1(\theta_3 - \theta_2) - n_2 n_3(2\theta_1 - \theta_2 - \theta_3) - 2n_2^2(\theta_1 - \theta_3) - 2n_3^2(\theta_1 - \theta_2) = 0 \tag{18}$$

Provided that the shear-tensor is not degenerate (if this would be the case, then substituting e.g. $H_2 = H_1$ in (18) and its cyclic permutations shows that also H_{ab} is degenerate, such that the solution would be of Petrov type D), one can eliminate the coefficients n_α from equations (18) and (15) and their cyclic permutations, to obtain

$$H_{11}^2 + H_{11}H_{22} + H_{22}^2 = 0 \tag{19}$$

which is clearly inconsistent with the fact that the Petrov type is I.

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